				Modulus	angle
$a=+10$	-10	$+10i$	$-10i$	Modulus	angle
$A=+450$	$-1{,}050$	$+300+750i$	$+300-750i$	$\sqrt{652{,}500}$	$68°\ 11'\ 55''$
$A_1=+4{,}015$	$-3{,}865$	$+75-4{,}060i$	$+75+4{,}060i$	$\sqrt{16{,}489{,}225}$	$-(88°\ 56'\ 30'')$
$A_2=+1{,}194$	$+1{,}194$	$-1{,}206$	$-1{,}206$	$1{,}206$	$180°\ 0'\ 0''$
$A_3=+240$	-240	$+240i$	$-240i$	240	$90°\ 0'\ 0''$
$A_4=+24$	$+24$	$+24$	$+24$	24	$0°\ 0'\ 0''$

Substituting in the terms of (2)

	$a=+10.00000$	-10.00000	$+10i$	$-10i$
$-AA_1^{-1}$	$=-0.11208$	-0.27167	$+0.18330-0.07728i$	$+0.18330+0.07728i$
$-\frac{1}{2}A^2A_2A_1^{-3}$	$=-0.00186$	$+0.01140$	$+0.00428+0.00402i$	$+0.00428-0.00402i$
$-\frac{1}{2}A^3A_2^2A_1^{-5}$	$=-0.00006$	-0.00095	$-0.00012-0.00033i$	$-0.00012+0.00033i$
$+\frac{1}{6}A^3A_3A_1^{-4}$	$=+0.00001$	$+0.00021$	$0.00000 \cdot 0.00000$	$0.00000\quad 0.00000$
$x=$ sum	$=+9.88601$	-10.26101	$+0.18746+9.92641i$	$+0.18746-9.92641i$

Should still greater accuracy be required replace the above values of a by the values of x just found, and so on until the required degree of accuracy is attained.

A FORMULA FOR THE SUM OF A CERTAIN TYPE OF INFINITE POWER SERIES.[1]

By ELBERT H. CLARKE, Purdue University.

INTRODUCTION.

The problem to be considered is that of finding a definite, finite formula which will give the sum of any convergent infinite series whose terms are such that their numerators form an arithmetical progression of any order[2] and whose denominators form a geometric progression.

Since the nth term of an arithmetic progression of the kth order may be reduced to the form

$$a_n = b_0n^k + b_1n^{k-1} + \cdots + b_k,$$

our problem is to evaluate the expression

$$T = \sum_{n=1}^{\infty} \frac{b_0n^k + b_1n^{k-1} + \cdots + b_k}{ar^n}$$

in which the b's are independent of n. But this may be written

$$T = \frac{b_0}{a}\sum_{n=1}^{\infty}\frac{n^k}{r^n} + \frac{b_1}{a}\sum_{n=1}^{\infty}\frac{n^{k-1}}{r^n} + \cdots + \frac{b_k}{a}\sum_{n=1}^{\infty}\frac{1}{r^n}$$

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[1] The author wishes to acknowledge criticisms and suggestions from Professors R. D. Carmichael and A. C. Lunn.

[2] See *Text Book of Algebra*, CHRYSTALL, Vol. 1, page 484.

and so the problem is at once reduced to that of finding a formula for

$$\sum_{n=1}^{\infty} \frac{n^k}{r^n} \qquad (|r| > 1,\ k \text{ a positive integer or zero})$$

which will be denoted in what follows by $S_{k,\,r}$.

The Form of the Sum.

The series $S_{k,\,r}$, being a power series in $1/r$, may be differentiated term by term with respect to r in its region of convergence.

$$\frac{dS_{k,\,r}}{dr} = \sum_{n=1}^{\infty} - \frac{n^{k+1}}{r^{n+1}} = -\frac{1}{r}\sum_{n=1}^{\infty} \frac{n^{k+1}}{r^n}.$$

This gives at once a fundamental relation,

$$S_{k+1,\,r} = -r\frac{dS_{k,\,r}}{dr}. \tag{1}$$

Furthermore, $S_{0,\,r}$ is none other than the ordinary geometric series, so we have at once

$$S_{0,\,r} = \frac{1}{r-1}$$

and we can derive any particular $S_{k,\,r}$ by k applications of the fundamental relation (1).

By an inspection of the forms of $S_{1,\,r}$; $S_{2,\,r} \cdots$; we are led to expect

$$S_{k,\,r} = \frac{r}{(r-1)^{k+1}} F_{k-1}(r),$$

where $F_{k-1}(r)$ is a polynomial of degree $(k-1)$ in r. Applying (1) we obtain, after a few simple reductions,

$$S_{k+1,\,r} = \frac{r}{(r-1)^{k+2}}\left[(kr+1)F_{k-1}(r) + (r-r^2)F'_{k-1}(r)\right]. \tag{2}$$

If $F_{k-1}(r)$ is a polynomial of degree $(k-1)$ in r then the bracketed quantity is a polynomial of degree k in r and may be called $F_k(r)$. In this way the form of $S_{k,\,r}$ is determined except for the particular coefficients occurring in the polynomial $F_{k-1}(r)$.

Determination of the Coefficients.

Let us write

$$F_{k-1}(r) = \alpha_{k-1,\,1}\, r^{k-1} + \alpha_{k-1,\,2}\, r^{k-2} + \alpha_{k-1,\,3}\, r^{k-3} + \cdots + \alpha_{k-1,\,k-1}\, r + \alpha_{k-1,\,k}.$$

Then by carrying out the operations indicated inside the brackets in (2) we obtain

$$\begin{aligned} F_k(r) = {} & \alpha_{k-1,\,1}\, r^k + [2\alpha_{k-1,\,2} + k\alpha_{k-1,\,1}]r^{k-1} \\ & + \cdots + [t\alpha_{k-1,\,t} + (k-t+2)\alpha_{k-1,\,t-1}]r^{k-t+1} \\ & + \cdots + [k\alpha_{k-1,\,2} + 2\alpha_{k-1,\,k-1}]r + \alpha_{k-1,\,k}. \end{aligned} \tag{3}$$

The following relation is seen to exist between the coefficients of $F_k(r)$ and $F_{k-1}(r)$:

$$\alpha_{k,\,t} = t\alpha_{k-1,\,t} + (k - t + 2)\alpha_{k-1,\,t-1}.$$

By inspection of $F_1(r)$, $F_2(r)$ $\cdots$ we notice that the coefficients are symmetrically arranged and that the first and last coefficients are always 1. By our work we see that the first and last coefficients in $F_{k-1}(r)$ and $F_k(r)$ are the same, therefore they must be 1. Let us assume further a symmetrical arrangement of the coefficients in $F_{k-1}(r)$. That is to say $\alpha_{k-1,\,t} = \alpha_{k-1,\,k-t+1}$. This assumption, in connection with the above recurrence relation between the coefficients of $F_k(r)$ and $F_{k-1}(r)$, gives us immediately

$$\begin{aligned}\alpha_{k,\,t} = t\alpha_{k-1,\,t} + (k - t + 2)\alpha_{k-1,\,t-1} = (k - t + 2)\alpha_{k-1,\,k-t+2} \\ + t\alpha_{k-1,\,k-t+1} = \alpha_{k,\,k-t+2}.\end{aligned} \tag{4}$$

So that symmetry of arrangement of the coefficients in $F_{k-1}(r)$ implies symmetry of arrangement of the coefficients in $F_k(r)$. That the coefficients in $F_k(r)$ are symmetrically arranged follows at once from an inspection of a particular $F_k(r)$, e. g., $F_4(r) = r^4 + 26r^3 + 66r^2 + 26r + 1$.

It is an interesting fact that the sum of the coefficients in $F_k(r)$ is $(k + 1)!$. The reader may easily derive the proof from (3).

As already stated, the *first* coefficient satisfies the relation

$$\alpha_{k,\,1} = \alpha_{k-1,\,1} = \cdots = 1.$$

By carrying out the recurrence relations for the *second* coefficient we obtain

$$\begin{aligned}
\alpha_{2,\,2} &= 2\alpha_{2,\,1} + 2, \qquad\qquad (\alpha_{2,\,1} = 1)\\
\alpha_{3,\,2} &= 2^2 \cdot 1 + 2 \cdot 2 + 3,\\
\alpha_{4,\,2} &= 2^3 \cdot 1 + 2^2 \cdot 2 + 2 \cdot 3 + 4,\\
&\cdots\cdots\cdots\cdots\\
\alpha_{k,\,2} &= 2^{k-1} \cdot 1 + 2^{k-2} \cdot 2 + 2^{k-3} \cdot 3 + \cdots + 2^2(k - 2) + 2(k - 1) + k.
\end{aligned}$$

If we write $\alpha_{k,\,2}$ in rows and add by columns we shall obtain without difficulty

$$\alpha_{k,\,2} = 2^{k+1} - (k + 2).$$

To determine the *third* coefficient we may proceed as above making use of the following formulas:

$$\begin{aligned}
\sum_{k=0}^{e} 3^{e-k} \cdot 2^k &= 3^{e+1} - 2^{e+1},\\
\sum_{k=0}^{e} 3^{e-k} \cdot k^2 &= \sum_{k=0}^{e} 3^{e-k} \cdot k + \tfrac{1}{2} \sum_{k=0}^{e} 3^{e-k} - \tfrac{1}{2}(e + 1)^2,\\
\sum_{k=0}^{e} 3^{e-k} \cdot k &= \tfrac{1}{2} \sum_{k=0}^{e} 3^{e-k} - \tfrac{1}{2}(e + 1).
\end{aligned}$$

When we have gone through with the necessary work, which is not difficult but is tedious, we obtain the following result

$$\begin{aligned}\alpha_{k,3} &= 2^2[3^{k-2}\cdot 1 + 3^{k-3}\cdot 2\cdot 2 + \cdots + 3(k-2)2^{k-3} + (k-1)2^{k-2}] \\ &\qquad -[3^{k-2}\cdot 1\cdot 3 + 3^{k-3}\cdot 2\cdot 4 + \cdots + 3(k-2)k + (k-1)(k+1)] \\ &= 3^{k+1} - 2^{k+1}(k+2) + \frac{(k+2)(k+1)}{2}\,.\end{aligned}$$

It is quite possible, of course, to proceed in the same way to obtain the fourth coefficient and so on. But the work will be extremely tedious and will not be likely to render much easier the generalization by inspection which must be made in any case. By an inspection of the results for $\alpha_{k,\,2}$ and for $\alpha_{k,\,3}$, together with (3) we are led to assume for the tth coefficient in $F_k(r)$,

$$\begin{aligned}\alpha_{k,\,t} = \Bigg[t^{k+1} - (t-1)^{k+1}\binom{k+2}{1} + \cdots + (-1)^{s-1}(t-s+1)^{k+1}\binom{k+2}{s-1} \\ + \cdots + (-1)^{t-1}\binom{k+2}{t-1}\Bigg],\end{aligned}$$

which gives the proper forms for $\alpha_{k,\,2}$ and $\alpha_{k,\,3}$ above and which further yields proper numerical values for $\alpha_{1,\,4}$, $\alpha_{2,\,4}$, $\cdots$. All that is needed to complete the induction is a proof that $\alpha_{k,\,t}$ will satisfy the recurrence relation (4). That is to say,

$$\begin{aligned}t^{k+1} - (t-1)^{k+1}\binom{k+2}{1} + \cdots + (-1)^{s-1}(t-s+1)^{k+1}\binom{k+2}{s-1} + \cdots \\ + (-1)^{t-2}2^{k+1}\binom{k+2}{t-2} + (-1)^{t-1}\binom{k+2}{t-1}\end{aligned}$$

must be equal to

$$\begin{aligned}t\Bigg[t^{k} - (t-1)^{k}\binom{k+1}{1} + \cdots + (-1)^{s-1}(t-s+1)^{k}\binom{k+1}{s-1} + \cdots \\ + (-1)^{t-1}\binom{k+1}{t-1}\Bigg] + (k-t+2)\Bigg[(t-1)^k - (t-2)^k\binom{k+1}{1} + \cdots \\ + (-1)^{s-1}(t-s)^k\binom{k+1}{s-1} + \cdots + (-1)^{t-2}\binom{k+1}{t-2}\Bigg].\end{aligned}$$

Carrying out the work, the latter expression becomes

$$\begin{aligned}t^{k+1} - (t-1)^k t\binom{k+1}{1} + \cdots + (1)^{s-1}(t-s+1)^k t\binom{k+1}{s-1} + \cdots + (-1)^{t-1}t\binom{k+1}{t-1} \\ + (t-1)^k(k-t+2) + \cdots + (-1)^{s-2}(t-s+1)^k(k-t+2)\binom{k+1}{s-2} + \cdots \\ + (-1)^{t-2}(k-t+2)\binom{k+1}{t-2}.\end{aligned}$$

Combining like terms, we have

$$t^{k+1}-(t-1)^k[t(k+1)-k+t-2]+\cdots+(-1)^{s-1}(t-s+1)^k\binom{k+1}{s-2}$$
$$\times\left[t\frac{k-s+3}{s-1}-k+t-2\right]+\cdots+(-1)^{t-1}\binom{k+1}{t-2}\left[t\frac{k-t+3}{t-1}-k+t-2\right],$$

which reduces to the left member above:

$$t^{k+1}-(t-1)^{k+1}\binom{k+2}{1}+\cdots+(-1)^{s-1}(t-s+1)^{k+1}\binom{k+2}{s-1}+\cdots$$
$$+(-1)^{t-1}\binom{k+2}{t-1}.$$

If we replace k by $k-1$ we shall have the tth term in $F_{k-1}(r)$. Hence we have the desired formula for $S_{k,r}$:

$$S_{k,r}=\sum_{n=1}^{\infty}\frac{n^k}{r^n}=\frac{r}{(r-1)^{k+1}}\Bigg\{r^{k-1}+[2^k-(k+1)]r^{k-2}+\cdots$$
$$+\left[t^k+\cdots+(-1)^{s-1}(t-s+1)^k\binom{k+1}{s-1}+\cdots\right.$$
$$\left.+(-1)^{t-1}\binom{k+1}{t-1}\right]r^{k-t}+\cdots+1\Bigg\}.$$

The general series of the introductory paragraph may now be written

$$T=\frac{b_0}{a}S_{k,r}+\frac{b_1}{a}S_{k-1,r}+\cdots+\frac{b_r}{a}S_{0,r}.$$

BOOK REVIEWS.

Memorabilia Mathematica. By ROBERT EDOUARD MORITZ. vii+410 pages. The Macmillan Co., New York, 1914. \$3.00 net.

In the Memorabilia Mathematica we are presented with a collection of more than 1100 quotations pertaining to many phases of mathematics and to the life and thought of mathematicians. The page preceding the title page is graced by two quotations from Goethe and Emerson. They read: "Alles Gescheite ist schon gedacht worden; man muss nur versuchen es noch einmal zu denken," and "A great man quotes bravely, and will not draw on his own invention when his memory serves him with a word as good."

We may safely infer that in these quotations, if not interpreted too literally, the author expresses a conviction and implies a purpose. The intrinsic value of the ideas expressed in the Memorabilia shows that the author's conviction is well founded. His purpose is realized in bringing this wealth of ideas within easy reach of mathematical and non-mathematical students.